Exercise 1. 1. Since $f(0)=0$, the continuous function

$$f^*(z)=\begin{cases}\dfrac{f(z)}{z} & \text{for all } z\in\mathbb{D}\setminus\{0\}\\ f'(0) & \text{for } z=0\end{cases}$$

is homomorphic on $\mathbb{D}$ (this follows simply by expanding f in Laurent series). By the maximum principle, for all $0<r<1$, we have

$$\sup_{z\in\mathbb{D}(0,r)}|f^*(z)|=\sup_{|z|=r}|f^*(z)|\leq\frac{1}{r}$$

as $|f(z)|\leq 1$. Let $r\to 1$, we deduce that $|f^*(z)|\leq 1$ for all $z\in\mathbb{D}$, which shows that $|f(z)|\leq|z|$ for all $z\in\mathbb{D}$. If equality holds, we deduce by the maximum principle that f^* is a constant function, which implies that $f(z)=az$ for some $a\in S^1$.

2. Indeed, we have

$$\begin{aligned}|z-a|^2-|1-\overline{a}z|^2&=|z|^2-2\operatorname{Re}(\overline{a}z)+|a|^2-\left(1-2\operatorname{Re}(\overline{a}z)+|a|^2|z|^2\right)\\&=|a|^2(1-|z|^2)+|z|^2-1=(1-|z|^2)(|a|^2-1)<0\end{aligned}$$

since $|a|<1$. Furthermore, we easily show that $f_{0,a}$ is its own inverse, which implies the result.

3. Consider $f_{\varphi(0)}\circ\varphi$. Then, the Schwarz lemma implies that $f_{\varphi(0)}\circ\varphi(z)=e^{i\theta}z$ for some $\theta\in[0,2\pi[$, and since $f_{0,\varphi(0)}$ is its own inverse, we deduce that

$$f(z)=f_{0,\varphi(0)}\left(e^{i\theta}z\right)=\frac{e^{i\theta}z-f(0)}{1-\overline{f(0)}e^{i\theta}z}=e^{i\theta}\,\frac{z-e^{-i\theta}f(0)}{1-\overline{e^{-i\theta}f(0)}z}$$

and we can take $a=e^{-i\theta}f(0)\in\mathbb{D}$.

Exercise 2. Let

$$f=\frac{1+\left(\frac{\partial u}{\partial x}\right)^2}{\sqrt{1+|\nabla u|^2}},\quad g=\frac{\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}}{\sqrt{1+|\nabla u|^2}},\quad h=\frac{1+\left(\frac{\partial u}{\partial y}\right)^2}{\sqrt{1+|\nabla u|^2}}.$$

An easy computation using the minimal surface equation shows that $\dfrac{\partial f}{\partial y}=\dfrac{\partial g}{\partial x}$ and $\dfrac{\partial h}{\partial x}=\dfrac{\partial g}{\partial y}$. Then, the function

$$\varphi(x,y)=\int_0^x\int_0^y g(s,t)dtds+\int_0^x\int_0^t f(s,0)dsdt+\int_0^y\int_0^s h(0,t)dtds$$

yields the solution.

Exercise 3. 1. The regularity follows from the classical theorems of smooth dependence of integrals with respect to a parameter. Integrating by parts and using the parity of φ, we get for all $\psi\in\mathscr{D}(\Omega)$

$$\begin{aligned}\int_\Omega\Delta(\varphi_\varepsilon*u)\psi\,dx&=\int_\Omega\varphi_\varepsilon*u\,\Delta\psi=\int_\Omega u\varphi_**\Delta\psi\\&=\int_\Omega u\Delta(\varphi_**\psi)dx=0\end{aligned}$$

where the final equality follows from the identity $D(f*g)=Df*g=f*(Dg)$ for all linear differential operator D and smooth functions f and g (as long as the integrals converge).

2. This is classical and omitted.

3. First, it is easy to establish that u also satisfies the mean value formula on spheres by differentiating the identity. In other words, we have

$$u(x) = \fint_{\partial B(x,r)} u(y)d\mathscr{H}^{d-1}(y) = \frac{1}{\beta(d)r^{d-1}}\int_{\partial B(x,r)} u(y)d\mathscr{H}^{d-1}(y).$$

Now, let $\psi \in \mathscr{D}(]0,1[)$ be a positive function such that

$$\int_0^1 r^{d-1}\psi(r)dr = \frac{1}{\beta(d)}.$$

Set $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(\varepsilon^{-1}|x|)$. For all $x \in \Omega_\varepsilon$, the function $y \mapsto \varphi_\varepsilon(x-y)$ has its support included in $\overline{\Omega_\varepsilon + B(0,\varepsilon)} \subset \Omega$ by definition of Ω_ε. Therefore, we have by the mean-value formula

$$\begin{aligned}\int_{\mathbb{R}^d} u(y)\varphi_\varepsilon(x-y)dy &= \int_{\mathbb{R}^d} u(x-y)\varphi_\varepsilon(y)dy = \frac{1}{\varepsilon^d}\int_{B(0,\varepsilon)} u(x-y)\psi(\varepsilon^{-1}|y|)dy \\ &= \int_{B(0,1)} u(x-\varepsilon z)\psi(|z|)dz = \int_0^1\int_{S^{d-1}} u(x-\varepsilon r\, y)\psi(r)r^{d-1}dr d\mathscr{H}^{d-1}(y) \\ &= \beta(d)u(x)\int_0^1 r^{d-1}\psi(r)dr = u(x)\end{aligned}$$

which shows that $u \in C^\infty(\Omega_\varepsilon)$. As the result is true for all $\varepsilon > 0$, the global regularity $u \in C^\infty(\Omega)$ follows.

4. Since $\varphi_\varepsilon * u$ converges almost everywhere to u as $\varepsilon \to 0$, we deduce that u satisfies the mean value formula. In particular, we have $u \in L^\infty_{\mathrm{loc}}(\Omega)$. Therefore, we get for all $x, y \in \Omega$ and $r > 0$ small enough

$$|u(x)-u(y)| = \frac{1}{\alpha(d)r^d}\left|\int_{B(x,r)} u(z)dz - \int_{B(y,r)} u(t)dt\right| \leq \frac{1}{\alpha(d)r^d}\int_{(B(x,r)\cup B(y,r))\setminus(B(x,r)\cap B(y,r))} |u(w)|dw$$

which shows the continuity of u since u is locally bounded.